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Translated by L.K.

PMM U.S.S.R.,Vol.48,No.6,pp.678-682,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

ON THE OPTIMAL CONTROL OF VISCOUS INCOMPRESSIBLE FLUID FLOW*

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The framework of the Navier-Stokes (N-S) equations is used to study flow past an arbitrary body on whose surface the tangential or normal velocity is under control. The necessary conditions are obtained for the minimum rate of energy dissipation. Exact analytical solutions of the corresponding problems are found for the case of flow past an ellipsoid in the Stokes approximation.

1. Let a body S by streamlined by a stationary flow of a viscous incompressible fluid. We shall consider the following variational problem: to find a suction (injection) velocity distribution over the body surface, for which the rate of energy dissipation D is minimal. We shall assume here that the total flow of fluid across the surface of S is zero.

Using dimensionless variables we write the equations of motion for the fluid, the boundary conditions and the minimizing functional in the form

$$\Delta \mathbf{V} - \nabla p - R(\mathbf{V}\cdot\nabla)\mathbf{V} = 0, \quad \nabla\cdot\mathbf{V} = 0, \quad \mathbf{V}|_S = Wn, \quad \mathbf{V}|_\infty = \mathbf{U} \tag{1.1}$$

$$D(W) = \int_\Omega \frac{1}{2} \sum_{i,j=1}^{3} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)^2 d\Omega \tag{1.2}$$

where Ω is the outside of the body S, $\mathbf{n} = (n_1, n_2, n_3)$ is the unit vector of the external normal, $\mathbf{U}$ is the stream velocity at infinity and R is the Reynolds number. The N-S equations are made dimensionless so as to ensure their simplest form in the limiting case of the Stokes flow as $R \to 0$.

2. To obtain the necessary condition for the minimum of the functional (1.2), we shall write the rate of suction (injection) W, the rate of flow $\mathbf{V}$ and the pressure p in the form

$$W = W_0 + \varepsilon W_1, \quad V = V_0 + \varepsilon V_1 + O(\varepsilon^2)$$
$$p = p_0 + \varepsilon p_1 + O(\varepsilon^2), \quad 0 < \varepsilon \ll 1 \tag{2.1}$$

The functions W_0, $\mathbf{V}_0$ and p_0 satisfy the boundary condition (1.1), while W_1, $\mathbf{V}_1$ and p_1 satisfy the boundary value problem

$$\Delta \mathbf{V}_1 - \nabla p_1 - R[(\mathbf{V}_0\cdot\nabla)\mathbf{V}_1 + (\mathbf{V}_1\cdot\nabla)\mathbf{V}_0] = 0, \quad \nabla\cdot\mathbf{V}_1 = 0$$
$$\mathbf{V}_1|_S = W_1\mathbf{n}, \quad \mathbf{V}_1|_\infty = 0 \tag{2.2}$$

Varying (1.2) and using the boundary conditions and Gauss's theorem, we obtain

$$\delta D = -2\varepsilon \int_\Omega \mathbf{V}_1\cdot\Delta\mathbf{V}_0\, d\Omega - 4\varepsilon \int_S \frac{\partial V_{n0}}{\partial n} W_1\, dS \tag{2.3}$$

**Prikl.Matem.Mekhan.*,48,6,929-934,1984

Let us scalar multiply the first equation of (2.2) by the function $\mathbf{V}^*$, so far arbitrary, and the second equation by the function p^*. Combining these expressions and integrating, we obtain the following relation:

$$\int_\Omega \{\mathbf{V}^*\cdot(\Delta\mathbf{V}_1 - \nabla p_1 - R[(\mathbf{V}_0\cdot\nabla)\mathbf{V}_1 + (\mathbf{V}_1\cdot\nabla)\mathbf{V}_0]) + p^*\nabla\cdot\mathbf{V}_1\}\,d\Omega = 0 \tag{2.4}$$

Let us now impose on $\mathbf{V}^*$ the homogeneous boundary conditions $\mathbf{V}^*|_\infty = \mathbf{V}^*|_\infty = 0$. Integrating (2.4) by parts we obtain

$$\int_\Omega \{\mathbf{V}_1\cdot(\Delta\mathbf{V}^* - \nabla p^* + R[(\mathbf{V}_0\cdot\nabla)\mathbf{V}^* - (\nabla\mathbf{V}_0)\cdot\mathbf{V}^*]) + p_1\nabla\cdot\mathbf{V}^*\}\,d\Omega + \int_S \left(\frac{\partial V_n^*}{\partial n} - p^*\right) W_1\,dS = 0 \tag{2.5}$$

where V_n^* is the component of the function $\mathbf{V}^*$ in the $\mathbf{n}$ direction. Let us obtain the functions $\mathbf{V}^*$ and p^* as a solution of the following boundary value problem:

$$\Delta\mathbf{V}^* - \nabla p^* + R[(\mathbf{V}_0\cdot\nabla)\mathbf{V}^* - (\nabla\mathbf{V}_0)\cdot\mathbf{V}^*] = 2\Delta\mathbf{V}_0,\quad \nabla\cdot\mathbf{V}^* = 0 \tag{2.6}$$
$$\mathbf{V}^*|_S = \mathbf{V}^*|_\infty = 0$$

The system of equations (2.6) is usually encountered when solving problems of optimization in a viscous incompressible fluid. In particular, the system was obtained in /1/. A detailed description of methods of solving optimal problems occurring in various branches of mechanics is given in /2/. From (2.5),(2.6) we find that

$$2\int_\Omega \mathbf{V}_1\cdot\Delta\mathbf{V}_0\,d\Omega = -\int_S \left(\frac{\partial V_n^*}{\partial n} - p^*\right) W_1\,dS$$

As a result, relation (2.3) takes the final form

$$\delta D = \varepsilon\int_S \left(\frac{\partial V_n^*}{\partial n} - p^* - 4\frac{\partial V_{n0}}{\partial n}\right) W_1\,dS \tag{2.7}$$

Since the minimum dissipation rate is sought under the condition that the total flux of fluid across the surface of S is zero, the function W_1 satisfies the condition

$$\int_S W_1\,dS = 0 \tag{2.8}$$

From (2.7),(2.8) we obtain the necessary condition for the functional $D(W)$ to be extremal

$$\left(\frac{\partial V_n^*}{\partial n} - p^* - 4\frac{\partial V_{n0}}{\partial n}\right)\Big|_S = \text{const} \tag{2.9}$$

3. Let us now consider the Stokes approximation $(R \to 0)$. In this case, as was shown in /3/, the boundary value problem (2.6) has a trivial solution $\mathbf{V}^* = 0$, $p^* = -2p_0 + \text{const}$ only. Therefore, the necessary condition for the dissipation rate to be a minimum takes the relatively simple form

$$\left(p_0 - 2\frac{\partial V_{n0}}{\partial n}\right)\Big|_S = \text{const} \tag{3.1}$$

As an example, we shall find the optimal distribution of the rate of suction (injection) over the surface of a sphere of unit radius, whose centre lies at the origin of a spherical system of coordinates (r, θ, φ). From considerations of symmetry and the linear nature of the problem, we seek the solution in the form

$$\mathbf{V} = \mathbf{U} + \operatorname{rot}\operatorname{rot}(f\mathbf{U}),\ f = f(r) \tag{3.2}$$

A more detailed explanation justifying the representation of the solution in the form (3.2) was given in /4/, using the Stokes problem of the flow past a sphere (without suction). The solution $f = Ar + B/r$ was obtained in /4/ where the constants A and B were found from the boundary conditions. Thus the velocity and pressure are given by the formulas

$$\mathbf{V} = \mathbf{U} - A\frac{\mathbf{U} + \mathbf{n}(\mathbf{U}\cdot\mathbf{n})}{r} + B\frac{3\mathbf{n}(\mathbf{U}\cdot\mathbf{n}) - \mathbf{U}}{r^3},\qquad p = p_\infty - \frac{2A(\mathbf{U}\cdot\mathbf{n})}{r^2} \tag{3.3}$$

The boundary condition at the sphere surface has, in the present case, the form

$$\mathbf{U} - A[\mathbf{U} + \mathbf{n}(\mathbf{U}\cdot\mathbf{n})] + B[3\mathbf{n}(\mathbf{U}\cdot\mathbf{n}) - \mathbf{U}] = W\mathbf{n}$$

This is possible only when the following conditions hold:

$$1 - A - B = 0, \quad (3B - A)(\mathbf{U}\cdot\mathbf{n}) = W \tag{3.4}$$

From (3.1),(3.3) and (3.4) we obtain the final expression for the optimal suction (injection) rate at the sphere surface

$$W = \frac{1}{3}(\mathbf{U}\cdot\mathbf{n}) = \frac{1}{3}U\cos\theta \tag{3.5}$$

Restoring the original dimensions, we find that the rate of dissipation D and the drag X of the sphere in this case are

$$D = \frac{16\pi}{3}\eta a U^2, \quad X = \frac{16\pi}{3}\eta a U \tag{3.6}$$

where a is the radius of the sphere and η is the coefficient of viscosity.

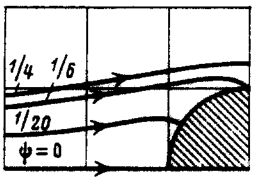

Fig. 1

Comparing this with the flow past an impermeable sphere we find, that the dissipation and drag were both reduced by about 11%. We note that the optimal solution obtained yields the absolute minimum for the dissipation rate for all viscous flows past a sphere with suction (injection). This follows from the Helmholtz variational principle /5/ according to which the rate of dissipation in a flow described by the N-S equations cannot be less that the rate of dissipation in a Stokes flow with the same boundary conditions.

Figure 1 shows the streamline pattern in a flow past a sphere when the suction (injection) rate is optimal. We note that when the flux rate is given $Q \neq 0$, all the results of Sect.2 still hold. In the special case of Stokes flow past a sphere, the solution sought will be the sum of the solution with zero flux, and the source situated at the centre of the sphere, with flow rate Q.

To realize optimal suction (injection), we must shed some power

$$N = \int_S W^2\, dS \tag{3.7}$$

All the results obtained in Sect.2 can be generalized to the case when the constraint (3.7) is present. The necessary condition of optimality (2.9) will now take the form

$$\left(\frac{\partial V_n^*}{\partial n} - p^* - 4\frac{\partial V_{n0}}{\partial n} - 2\nu W\right)\Big|_S = \text{const} \tag{3.8}$$

where ν is the Lagrange multiplier corresponding to the power constraint imposed on the control. In the Stokes approximation, condition (3.8) takes the form

$$\left(p_0 - 2\frac{\partial V_{n0}}{\partial n}\right)\Big|_S = \nu W + \text{const} \tag{3.9}$$

We will seek the solution of the optimal problem again in the form (3.2),(3.3) The conditions of optimality (3.9) and the boundary conditions together yield

$$\nu(3B - A) + 2A + 2(2A - 6B) = 0 \tag{3.10}$$

The constraint imposed on the power N yields the last relation for determining A, B and ν

$$\frac{4\pi}{3}(3B - A)^2 = \frac{N}{U^2} \tag{3.11}$$

From (3.4), (3.10) and (3.11) we obtain

$$A = \frac{3}{4}\left(1 - \sqrt{\frac{N}{12\pi U^2}}\right)$$

Thus the optimal solution in this case has the form

$$W = \sqrt{\frac{3N}{4\pi}}\cos\theta \tag{3.12}$$

$$D = 6\pi\eta a U^2\left(1 - \sqrt{\frac{N_*}{3}} + \frac{3N_*}{4}\right)$$

$$X = 6\pi\eta a U\left(1 - \sqrt{\frac{N_*}{12}}\right); \quad N_* = \frac{N}{\pi U^2}$$

4. We shall consider an ellipsoid with semiaxes a, b and c, as the example illustrating the exact solution of the optimizing problem for a non-axially symmetric body. In solving the problem we shall follow the solution of the Stokes flow past an ellipsoid without suction (injection) given in /6/. We will seek the solution $\mathbf{V} = (u, v, w)$ in the form

$$u = A\frac{\partial^2\Phi}{\partial x^2} + B\left(x\frac{\partial\chi}{\partial x} - \chi\right) + U \tag{4.1}$$

$$v = A\frac{\partial^2\Phi}{\partial x\,\partial y} + Bx\frac{\partial\chi}{\partial y}, \quad w = A\frac{\partial^2\Phi}{\partial x\,\partial z} + Bx\frac{\partial\chi}{\partial z}$$

$$\Phi = \pi abc\int\limits_\lambda^\infty\left(\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} - 1\right)\frac{d\mu}{F}$$

$$\chi = abc\int\limits_\lambda^\infty\frac{d\mu}{F}, \quad F = [(a^2+\mu)(b^2+\mu)(c^2+\mu)]^{1/2}$$

Here λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$$

and A and B are constants, so far unknown. The pressure is given by the formula $p = 2B\partial\chi/\partial x + p_\infty$.

Taking into account equations (4.1) and the identity

$$\frac{\partial\Phi}{\partial x} = 2\pi\alpha x, \quad \alpha = \alpha(\lambda) = abc\int\limits_\lambda^\infty\frac{d\mu}{(a^2+\mu)F}$$

we write the boundary conditions $u = Wn_1,\ v = Wn_2, w = Wn_3$ at the surface of the ellipsoid in the form

$$2\pi A\alpha_0 + U + 2\pi A\left(x\frac{\partial\lambda}{\partial x}\frac{\partial\alpha}{d\lambda}\right)_{\lambda=0} + B\left(x\frac{\partial\lambda}{\partial x}\frac{d\chi}{d\lambda}\right)_{\lambda=0} - B\chi_0 = W\frac{x}{a^2E} \tag{4.2}$$

$$2\pi A\left(x\frac{\partial\lambda}{\partial y}\frac{d\alpha}{d\lambda}\right)_{\lambda=0} + B\left(x\frac{\partial\lambda}{\partial y}\frac{d\chi}{d\lambda}\right)_{\lambda=0} = W\frac{y}{b^2E} \tag{4.3}$$

$$2\pi A\left(x\frac{\partial\lambda}{\partial z}\frac{d\alpha}{d\lambda}\right)_{\lambda=0} + B\left(x\frac{\partial\lambda}{\partial z}\frac{d\chi}{d\lambda}\right)_{\lambda=0} = W\frac{z}{c^2E} \tag{4.4}$$

Here

$$\alpha_0 = \alpha(0), \quad \chi_0 = \chi(0), \quad E = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{1/2}$$

Taking into account the fact that at the ellipsoid surface

$$\frac{\partial\lambda}{\partial x} = \frac{2x}{a^2E^2}, \quad \frac{\partial\lambda}{\partial y} = \frac{2y}{b^2E^2}, \quad \frac{\partial\lambda}{\partial z} = \frac{2z}{c^2E^2}$$

we obtain, from the conditions (4.3) and (4.4).

$$W = -2\left(\frac{2\pi A}{a^2} + B\right)\frac{x}{E} \tag{4.5}$$

Substituting (4.5) into (4.2) we obtain $2\pi\alpha_0 A + U = B\chi_0$. The final formula for the optimal suction (injection) law is obtained after deriving the second equation connecting A and B from the optimality condition (3.1), and has the form

$$W = \frac{2U(d-1)}{d\alpha_0 a^2 + \chi_0}\,\frac{x}{E}, \quad d = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{-1} \tag{4.6}$$

The drag is given by the formula

$$X = 16\pi\eta U\frac{abc}{d\alpha_0 a^2 + \chi_0} \tag{4.7}$$

When $a = b = c$, which corresponds to flow past a sphere, expressions (4.6) and (4.7) become (3.5) and (3.6). We note that by virtue of the linearity of the Stokes equations we can obtain from (4.6) an explicit form of the solution of the optimal problem for the case of flow past an ellipsoid, when the direction of the flow velocity at infinity is arbitrary.

5. Let us consider the problem of minimum rate of dissipation of energy when the tangential velocity $\mathbf{q}$ at the boundary of the body S is controlled. Certain theoretical and experimental investigations of the viscous incompressible fluid flows with a moving boundary are described in /7/. Following exactly the case of controlling the normal velocity, we obtain the condition of optimality

$$\left[\left(\frac{\partial\mathbf{V}^*}{\partial n} - 2\frac{\partial\mathbf{V}_0}{\partial n}\right)\times\mathbf{n}\right]\Big|_S = 0 \tag{5.1}$$

Here $\mathbf{V}^*$ satisfies the boundary value problem (2.6), and $\mathbf{V}_0$ is the velocity in the optimal solution, the latter representing the solution of the boundary value problem (1.1) with another boundary condition at the surface $\mathbf{V}|_S = \mathbf{q}$. In the Stokes approximation the condition of optimality (5.1) becomes

$$\left[\frac{\partial \mathbf{V}_0}{\partial n} \times \mathbf{n}\right]\Big|_S = 0 \qquad (5.2)$$

The condition obtained means that, in fact, in the optimal solution the viscous tangential stresses at the body surface are zero.

Let us consider flow past a sphere with a moving boundary. We will again seek the solution of the problem in the form (3.3). From symmetry considerations in the spherical coordinate system we have $\mathbf{q} = (0, q, 0)$, and from the boundary conditions we have

$$A - B = \frac{1}{2}, \quad q = (1 - A - B)\, U \sin\theta \qquad (5.3)$$

The optimality condition (5.2) yields $A + 3B = 0$, and from (5.3) we obtain the final expression for the optimal solution (in dimensional coordinates)

$$q = \frac{3}{4} U \sin\theta, \quad D = \frac{3\pi}{2} \eta a U^2, \quad X = 3\pi\eta a U \qquad (5.4)$$

Comparison with the flow past a sphere with a fixed boundary shows that the rate of dissipation is reduced by 75%, and the drag by 50%. From the Helmholtz variational principle it follows that $D \geqslant {}^3/_2 \pi\eta a U^2$ for any viscous flow past a sphere with a moving boundary.

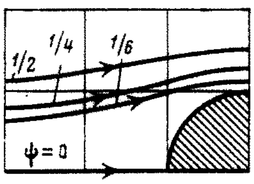

Fig. 2

Figure 2 shows the pattern of the streamlines around the sphere when the moving boundary is optimally controlled. When the power is under constraint

$$N = \int_S q^2\, dS$$

we obtain the following optimal solution:

$$q = \sqrt{\frac{3N}{8\pi}} \sin\theta, \quad D = 6\pi\eta a U^2 \left(1 - 3\sqrt{\frac{N_*}{6}} + \frac{N_*}{2}\right) \qquad (5.5)$$

$$X = 6\pi\eta a U \left(1 - \sqrt{\frac{N_*}{6}}\right); \quad N_* = \frac{N}{\pi U^2}$$

Following the method given in Sect.4, we can generalize the optimal solution (5.4) to the case of flow past a triaxial ellipsoid.

We note that when problems are solved without a constraint on N, in which the drag is minimized instead of the rate of dissipation, we find that an absolute minimum is attained by the solution with infinite thrust. In the case of flow past a sphere this result is reached at once from the formula $X = 8\pi\eta a U A$, so that when $A \to -\infty$, the drag $X \to -\infty$. When a constraint is imposed on N, the solution of the problem of the minimum has the same accuracy as the corresponding solutions (3.12) and (5.5) of the problems of minimum rate of dissipation.

The authors thank V.V. Sychev for valuable comments.

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